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Approximate Lie group analysis and solutions of 2D nonlinear diffusion–convection equations

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Abstract
Approximate Lie symmetries of the (2 + 1)-dimensional nonlinear diffusion equation with a small convection are completely classified. It is known that the invariance principle furnishes a systematic method of solving initial-value problems. The solutions of instantaneous source type of the 2D diffusion–convection equation are obtained for the case of power-law diffusivity, using a symmetry reduction.

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1. Introduction

The nonlinear diffusion–convection equation

$$u_t = \nabla(D(u)\nabla u) + K(u)\nabla u \tag{1}$$

has numerous applications in physics, chemistry and biology [1] (see a review of related references in [2, 3]). Much work has been devoted to constructing exact solutions of such equations [3–5]. One of the effective methods of solving nonlinear partial differential equations is the classical Lie group analysis [6–8]. The technique of Lie symmetry reduction provides relatively simple similarity solutions of equation (1) which are at the same time of significant physical interest. The group classification of one-dimensional equation (1) by point [9, 10] and potential [11] symmetries was performed. The symmetry properties of two- and three-dimensional equation (1) were classified in [3] and used there to find new exact solutions, in addition to the solutions obtained by Philip and Knight [5].

We consider (with first order of precision) two-dimensional nonlinear diffusion equation with a small convection

$$u_t = (\varphi(u)u_x)_x + (\psi(u)u_y)_y + \varepsilon f(u)u_x + \varepsilon g(u)u_y + o(\varepsilon) \tag{2}$$

where ε is a small parameter. In the filtration theory, for instance, ε characterizes the direction of unsaturated flow in porous media. We investigate the group properties of equation (2) by means of the theory of approximate Lie symmetries [12]. This gives advantages when some terms of an equation can be taken as being small. Often the group of approximate Lie symmetries of such an equation is larger than the group of classical Lie symmetries of the equation that provides more possibilities of finding exact solutions. Such is the case, for example, for equation

$$u_t = (u^\sigma u_x)_x + (u^\sigma u_y)_y + \varepsilon u^\sigma (u^\alpha + \beta) u_x + \varepsilon u^\sigma (u^\alpha + \gamma) u_y.$$

The only Lie symmetries are the time and space translations. While there is an additional approximate Lie symmetry for arbitrary β and γ (case 5 of section 2), an additional Lie symmetry, dilation, exists only if $\beta = 0$, $\gamma = 0$.

In section 2, the class of nonlinear equations (2) is completely classified with respect to admitted approximate Lie symmetries. The group properties of equation (2) turn out to be closely related to the properties of the pure diffusion equation

$$u_t = (\varphi(u) u_x)_x + (\psi(u) u_y)_y \tag{3}$$

analysed in [11]. It is a special case of the group classification result presented in [13] for the two- and three-dimensional heat equation with a source

$$u_t = \nabla(D(u)\nabla u) + Q(u). \tag{4}$$

Some exact solutions of equation (3) are found there for the isotropic case ($\varphi(u) = u^{-1}$, $\psi(u) = u^{-1}$) and a solution of 2D equation (4) with power-law forms of $D(u)$, $Q(u)$.

Here the use of approximate symmetries in solving initial-value problems allows us to reduce the number of independent variables of the problem. In section 3, in this way the solutions of instantaneous source type are found for equation (2) with the power-law form of the diffusion coefficient. The symmetry properties and group invariant solutions of the 1D diffusion equation with small convective terms were studied in [14].

2. Group classification of 2D nonlinear diffusion–convection equation

We perform an approximate Lie group analysis of equation (2) with an accuracy $o(\varepsilon)$. It means that hereinafter in all equalities we neglect the terms of order $o(\varepsilon)$.

We consider the infinitesimal operator

$$X = \tau(t,x,y,u,\varepsilon)\partial_t + \xi(t,x,y,u,\varepsilon)\partial_x + \eta(t,x,y,u,\varepsilon)\partial_y + \upsilon(t,x,y,u,\varepsilon)\partial_u \tag{5}$$

corresponding to the infinitesimal transformation of equation (2)

$$\begin{aligned} \bar{t} &= t + a\tau(t,x,y,u,\varepsilon) + o(a) & \bar{x} &= x + a\xi(t,x,y,u,\varepsilon) + o(a) \\ \bar{y} &= y + a\eta(t,x,y,u,\varepsilon) + o(a) & \bar{u} &= u + a\upsilon(t,x,y,u,\varepsilon) + o(a) \end{aligned} \tag{6}$$

where

$$\begin{aligned} \tau &= \tau^0(t,x,y,u) + \varepsilon\tau^1(t,x,y,u) + o(\varepsilon) & \xi &= \xi^0(t,x,y,u) + \varepsilon\xi^1(t,x,y,u) + o(\varepsilon) \\ \eta &= \eta^0(t,x,y,u) + \varepsilon\eta^1(t,x,y,u) + o(\varepsilon) & \upsilon &= \upsilon^0(t,x,y,u) + \varepsilon\upsilon^1(t,x,y,u) + o(\varepsilon). \end{aligned} \tag{7}$$

Operator (5) has the form $X = X^0 + \varepsilon X^1 + o(\varepsilon)$ with $X^i = \tau^i\partial_t + \xi^i\partial_x + \eta^i\partial_y + \upsilon^i\partial_u$, $i = 0, 1$, in this notation. The coordinates of operator (5) extended to first and second orders as

$$X_E = X + \upsilon_t\partial_{u_t} + \upsilon_x\partial_{u_x} + \upsilon_y\partial_{u_y} + \upsilon_{xx}\partial_{u_{xx}} + \upsilon_{yy}\partial_{u_{yy}}$$

are calculated by the prolongation formulae

$$\begin{aligned} \upsilon_{x_j} &= D_{x_j}\upsilon - u_{x_1}D_{x_j}\tau - u_{x_2}D_{x_j}\xi - u_{x_3}D_{x_j}\eta & j &= 1,2,3 \\ \upsilon_{x_ix_i} &= D_{x_i}\upsilon_{x_i} - u_{x_1x_i}D_{x_i}\tau - u_{x_2x_i}D_{x_i}\xi - u_{x_3x_i}D_{x_i}\eta & i &= 2,3 \end{aligned}$$

where $(x_1, x_2, x_3) = (t, x, y)$ and D_{x_i} is the total differentiation operator with respect to x_i. The condition

$$X_E(u_t - (\varphi(u)u_x)_x - (\psi(u)u_y)_y - \varepsilon f(u)u_x - \varepsilon g(u)u_y)|_{(2)} = o(\varepsilon) \qquad (8)$$

of invariance of equation (2) under the infinitesimal transformation (6) provides a set of determining equations, which are linear partial differential equations (PDEs) in functions (7). In order to analyse these equations we apply the equivalence transformations of equation (2)

$$\bar{t} = \alpha_1 t + \alpha_2 \qquad \bar{x} = \alpha_3 x + \varepsilon\alpha_4 t + \alpha_5 \qquad \bar{y} = \alpha_6 y + \varepsilon\alpha_7 t + \alpha_8 \qquad \bar{u} = \alpha_9 u + \alpha_{10}$$

$$\bar{\varphi} = \frac{\alpha_3^2}{\alpha_1}\varphi \qquad \bar{\psi} = \frac{\alpha_6^2}{\alpha_1}\psi \qquad \bar{f} = \frac{\alpha_3}{\alpha_1} f - \alpha_4 \qquad \bar{g} = \frac{\alpha_6}{\alpha_1} g - \alpha_7$$

$(\alpha_1, \alpha_9 > 0, \alpha_3, \alpha_6 \neq 0)$, which allow us to simplify the form of functions φ, ψ, f, g and do not alter the class of equations (2).

Splitting by powers of ε reduces (8) in zeroth order to the condition

$$X_E^0(u_t - (\varphi(u)u_x)_x - (\psi(u)u_y)_y)|_{(3)} = 0$$

of invariance of equation (3) under the infinitesimal transformation generated by X^0. In first order in ε we have the equality

$$X_E^1(u_t - (\varphi(u)u_x)_x - (\psi(u)u_y)_y)|_{(3)} + H = 0 \qquad (9)$$

where, given X^0, the addend H is calculated as

$$H = \frac{1}{\varepsilon} X_E^0(u_t - (\varphi(u)u_x)_x - (\psi(u)u_y)_y - \varepsilon f(u)u_x - \varepsilon g(u)u_y)|_{(2)}.$$

The symmetry properties of unperturbed equation (3) were studied in [13]. Therefore, for all functional forms of $\varphi(u)$ and $\psi(u)$ classified in [13], the terms $\tau^0, \xi^0, \eta^0, \upsilon^0$ of coordinates (7) of the infinitesimal operator (5) are known. For arbitrary $\varphi(u)$ and $\psi(u)$ the only symmetries of equation (3) are the translation and the dilation operators

$$Y_1 = \partial_t \qquad Y_2 = \partial_x \qquad Y_3 = \partial_y \qquad Y_4 = 2t\partial_t + x\partial_x + y\partial_y$$

and, if $\psi = \kappa\varphi, \kappa = \pm 1$, the rotation operator

$$Y_0 = y\partial_x - \kappa x\partial_y.$$

This implies $\tau^0 = C_1 + 2C_4 t, \xi^0 = C_2 + C_4 x + C_0 y, \eta^0 = C_3 + C_4 y - \kappa C_0 x, \upsilon^0 = 0$ and condition (9) yields the determining equations in $\tau^1, \xi^1, \eta^1, \upsilon^1$:

$$\tau^1 = \tau^1(t) \qquad \xi^1 = \xi^1(t, x, y) \qquad \eta^1 = \eta^1(t, x, y) \qquad \upsilon^1_{uu} = 0$$

$$\upsilon^1\varphi' + \left(\tau^1_t - 2\xi^1_x\right)\varphi = 0 \qquad \upsilon^1\psi' + \left(\tau^1_t - 2\eta^1_y\right)\psi = 0 \qquad \eta^1_x\varphi + \xi^1_y\psi = 0$$

$$\xi^1_t + \left(2\upsilon^1_{xu} - \xi^1_{xx}\right)\varphi - \xi^1_{yy}\psi + 2\upsilon^1_x\varphi' + C_4 f - C_0 g = 0 \qquad \upsilon^1_{xx}\varphi + \upsilon^1_{yy}\psi - \upsilon^1_t = 0$$

$$\eta^1_t - \eta^1_{xx}\varphi + \left(2\upsilon^1_{yu} - \eta^1_{yy}\right)\psi + 2\upsilon^1_y\psi' + C_4 g + \kappa C_0 f = 0 \qquad C_0(\kappa\varphi - \psi) = 0.$$

Hence for arbitrary φ, ψ, f, g we obtain $C_4 = 0, C_0 = 0$ and the *principal Lie algebra* $L_{\mathcal{P}}$ of equation (2) is spanned by operators

$$X_1 = \partial_t \qquad X_2 = \partial_x \qquad X_3 = \partial_y \qquad X_{3+i} = \varepsilon Y_i \qquad i = 1, 2, 3 \qquad X_7 = \varepsilon Y_4. \qquad (10)$$

For $\psi = \kappa\varphi, \kappa = \pm 1$, there is an additional operator

$$X_0 = \varepsilon Y_0. \qquad (11)$$

If, furthermore, $\kappa = -1$ and $g = f$, the basis of $L_{\mathcal{P}}$ is extended by the operator $Y_4 + Y_0$. These are the only symmetries admitted by equation (2) when $\varphi(u), \psi(u)$ do not enter (up to an equivalence transformation) into the list of functional forms which provide additional symmetries of equation (3).

Table 1. Symmetries for equation (2), $\varphi(u) = e^{\alpha u}$, $\psi(u) = \pm e^{\beta u}$.

$f(u)$	$g(u)$	Symmetries
$e^{\gamma u}$	$e^{(\gamma+(\beta-\alpha)/2)u}$ or 0	$(\alpha - 2\gamma)Y_4 + Y_5$
u	$e^{\frac{1}{2}(\beta-\alpha)u}$ or 0	$\alpha Y_4 + Y_5 - 2\varepsilon t\partial_x$

Table 2. Symmetries for equation (2), $\varphi(u) = e^u$, $\psi(u) = \kappa e^u$, $\kappa = \pm 1$.

$f(u)$	$g(u)$	κ	Symmetries
e^u	e^u	$\pm$	$Y_4 - Y_5$
			$Y_4 - \varepsilon Y_x - \varepsilon Y_y$
			$Y_0 - \varepsilon\kappa Y_y + \varepsilon Y_x$
ue^u	$(u + \alpha)\, e^u$	$\pm$	$\frac{1}{2}Y_4 - \frac{1}{2}Y_5 + \varepsilon Y_x + \varepsilon Y_y$
		$-$	$Y_4 + Y_0 + \varepsilon\alpha(Y_x - Y_y)$
ue^u	e^u or 0	$\pm$	$\frac{1}{2}Y_4 - \frac{1}{2}Y_5 + \varepsilon Y_x$
$(e^{\alpha u} + \beta)\, e^u$	$(e^{\alpha u} + \gamma)\, e^u$	$\pm$	$\frac{1}{2}Y_5 - \left(\alpha + \frac{1}{2}\right) Y_4 + \varepsilon\alpha(\beta Y_x + \gamma Y_y)$
		$-$	$Y_4 + Y_0 + \varepsilon(\gamma - \beta)(Y_x - Y_y)$
$(e^{\alpha u} + \beta)\, e^u$	γe^u	$\pm$	$\frac{1}{2}Y_5 - \left(\alpha + \frac{1}{2}\right) Y_4 + \varepsilon\alpha(\beta Y_x + \gamma Y_y)$
$u + \alpha e^u$	$u + \beta e^u$	$\pm$	$\frac{1}{2}Y_4 + \frac{1}{2}Y_5 - \varepsilon\alpha Y_x - \varepsilon\beta Y_y - \varepsilon t\partial_x - \varepsilon t\partial_y$
		$-$	$Y_4 + Y_0 + \varepsilon(\beta - \alpha)(Y_x - Y_y)$
$u + \alpha e^u$	e^u	$\pm$	$\frac{1}{2}Y_4 + \frac{1}{2}Y_5 - \varepsilon\alpha Y_x - \varepsilon t\partial_x - \varepsilon Y_y$
$e^u(e^{\alpha u}(\lambda\cos u + \mu\sin u) + \beta)$	$e^u(e^{\alpha u}(\mu\cos u - \lambda\sin u) + \gamma)$	$+$	$\frac{1}{2}Y_5 - \left(\alpha + \frac{1}{2}\right) Y_4 + Y_0 + \varepsilon a Y_x + \varepsilon b Y_y$, where $a = \alpha\beta + \gamma$, $b = \alpha\gamma - \beta$
$e^{\alpha u} + (\gamma u + \beta)\, e^u$	$e^{\alpha u} + (\gamma u - \beta)\, e^u$	$-$	$Y_4 + Y_0 - 2\varepsilon\beta(Y_x - Y_y)$
$(e^{\alpha u} + \beta u)\, e^u$	$(e^{\alpha u} - \beta u)\, e^u$	$-$	$Y_5 - (\alpha + 1)Y_4 + \alpha Y_0 - 2\varepsilon\beta(Y_x - Y_y)$
$(e^{\alpha u} + \gamma e^{\beta u} + \lambda)\, e^u$	$(e^{\alpha u} - \gamma e^{\beta u} + \mu)\, e^u$	$-$	$Y_5 + (\alpha - \beta)Y_0 - (\alpha + \beta + 1)Y_4 + \varepsilon a Y_x + \varepsilon b Y_y$, where $a = \lambda(\alpha + \beta) + \mu(\alpha - \beta)$, $b = \lambda(\alpha - \beta) + \mu(\alpha + \beta)$
$e^{\alpha u}$	0	$\pm$	$(1 - 2\alpha)Y_4 + Y_5$
e^u	0	$\pm$	$Y_4 - Y_5$
			$Y_4 - \varepsilon Y_x$
			$Y_0 - \varepsilon\kappa Y_y$
$u + \alpha e^u$	0	$\pm$	$\frac{1}{2}Y_4 + \frac{1}{2}Y_5 - \varepsilon\alpha Y_x - \varepsilon t\partial_x$

Similarly, proceeding with other kinds of $\varphi(u)$ and $\psi(u)$ from [13], we find the Lie algebra of admitted symmetries of corresponding equation (2) for arbitrary $f(u)$, $g(u)$ and then identify the functional forms of f and g which possess extra symmetries.

Case 1. $\varphi(u) = e^{\alpha u}$, $\psi(u) = \pm e^{\beta u}$. For arbitrary functions f and g the Lie algebra is spanned by operators (10) and the operator

$$X_8 = \varepsilon Y_5 \qquad \text{where} \quad Y_5 = \alpha x\partial_x + \beta y\partial_y + 2\partial_u.$$

Additional symmetries are presented in table 1.

Case 2. $\varphi(u) = e^u$, $\psi(u) = \kappa\, e^u$, $\kappa = \pm 1$. If f and g are arbitrary, the Lie algebra is spanned by operators (10), (11) and the operator

$$X_8 = \varepsilon Y_5 \qquad \text{where} \quad Y_5 = x\partial_x + y\partial_y + 2\partial_u.$$

In the classification result given in table 2 we use auxiliary operators

$$Y_x = \tfrac{1}{8}(x^2 - \kappa y^2)\partial_x + \tfrac{1}{4}xy\partial_y + \tfrac{1}{2}x\partial_u \qquad Y_y = \tfrac{1}{4}\kappa xy\partial_x + \tfrac{1}{8}(\kappa y^2 - x^2)\partial_y + \tfrac{1}{2}\kappa y\partial_u.$$

Table 3. Symmetries for equation (2), $\varphi(u) = \mathrm{e}^u$, $\psi(u) = \pm 1$.

$f(u)$	$g(u)$	Symmetries
$u\mathrm{e}^u$	$\mathrm{e}^{u/2}$ or 0	$Y_4 - Y_5 + 2\varepsilon Y_x$
$u + \alpha\mathrm{e}^u$	$\mathrm{e}^{-u/2}$ or 0	$Y_4 + Y_5 - 2\varepsilon\alpha Y_x - 2\varepsilon t\partial_x$
$(\mathrm{e}^{\beta u} + \alpha)\,\mathrm{e}^u$	$\mathrm{e}^{(\beta+1/2)u}$ or 0	$(1+2\beta)Y_4 - Y_5 - 2\varepsilon\alpha\beta Y_x$
$\mathrm{e}^{u/2} + \alpha\mathrm{e}^u$	u	$Y_5 - \varepsilon\alpha Y_x - 2\varepsilon t\partial_y$
e^u	$\mathrm{e}^{\alpha u}$	$2\alpha Y_4 - Y_5 + \varepsilon(1-2\alpha)Y_x$
e^u	0	$Y_5 - \varepsilon Y_x$
		$Y_4 - Y_5$

Table 4. Symmetries for equation (2), $\varphi(u) = u^\sigma$, $\psi(u) = \pm u^\rho$.

$f(u)$	$g(u)$	Symmetries
$u^{\alpha+\sigma/2}$	$u^{\alpha+\rho/2}$ or 0	$Y_5 - 2\alpha Y_4$
$\ln u$	$u^{(\rho-\sigma)/2}$ or 0	$\sigma Y_4 + Y_5 - 2\varepsilon t\partial_x$

Case 3. $\varphi(u) = \mathrm{e}^u$, $\psi(u) = \pm 1$. If f and g are arbitrary, the basis of Lie algebra is given by operators (10) and

$$X_8 = \varepsilon Y_5 \qquad \text{where} \quad Y_5 = x\partial_x + 2\partial_u.$$

In table 3 the cases of its extension are given. An auxiliary operator $Y_x = \frac{1}{6}x^2\partial_x + \frac{2}{3}x\partial_u$ is used there.

Case 4. $\varphi(u) = u^\sigma$, $\psi(u) = \pm u^\rho$. For arbitrary functions f and g the Lie algebra is spanned by operators (10) and the operator

$$X_8 = \varepsilon Y_5 \qquad \text{where} \quad Y_5 = \sigma x\partial_x + \rho y\partial_y + 2u\partial_u.$$

Additional symmetries are presented in table 4.

Case 5. $\varphi(u) = u^\sigma$, $\psi(u) = \kappa u^\sigma$, $\kappa = \pm 1$. For arbitrary functions f and g the Lie algebra is spanned by operators (10), (11) and

$$X_8 = \varepsilon Y_5 \qquad \text{where} \quad Y_5 = \sigma x\partial_x + \sigma y\partial_y + 2u\partial_u.$$

In table 5 additional symmetries are summarized. Here we use the notation

$$Y_x = \frac{\sigma}{8(\sigma+1)}((x^2 - \kappa y^2)\partial_x + 2xy\partial_y) + \frac{xu}{2(\sigma+1)}\partial_u$$
$$Y_y = \frac{\sigma}{8(\sigma+1)}(2\kappa xy\partial_x + (\kappa y^2 - x^2)\partial_y) + \frac{\kappa yu}{2(\sigma+1)}\partial_u.$$

Case 6. $\varphi(u) = u^{-1}$, $\psi(u) = \kappa u^{-1}$, $\kappa = \pm 1$. For arbitrary functions f and g the Lie algebra is infinite dimensional. It is generated by operators (10) and

$$X_\infty = \varepsilon(A(x, y)\partial_x + B(x, y)\partial_y - 2A_x u\partial_u)$$

where $A(x, y)$ and $B(x, y)$ are any solutions of the system $A_x = B_y, A_y = -\kappa B_x$. In table 6 the cases of extension of the basis of Lie algebra are given. An auxiliary operator is $Y_t = t\partial_t + u\partial_u$.

Table 5. Symmetries for equation (2), $\varphi(u) = u^\sigma$, $\psi(u) = \kappa u^\sigma$, $\kappa = \pm 1$.

$f(u)$	$g(u)$	κ	Symmetries
u^σ	u^σ	$\pm$	$\sigma Y_4 - Y_5$
			$Y_4 - \varepsilon Y_x - \varepsilon Y_y$
			$Y_0 - \varepsilon\kappa Y_y + \varepsilon Y_x$
$u^\sigma(u^\alpha + \beta)$	$u^\sigma(u^\alpha + \gamma)$	$\pm$	$\frac{1}{2}Y_5 - \left(\alpha + \frac{\sigma}{2}\right)Y_4 + \varepsilon\alpha(\beta Y_x + \gamma Y_y)$
		$-$	$Y_4 + Y_0 + \varepsilon(\gamma - \beta)(Y_x - Y_y)$
$\ln u + \alpha u^\sigma$	$\gamma \ln u + \beta u^\sigma$	$\pm$	$\frac{\sigma}{2}Y_4 + \frac{1}{2}Y_5 - \varepsilon t\partial_x - \varepsilon\sigma(\alpha Y_x + \beta Y_y) - \varepsilon\gamma t\partial_y$
	$\gamma = 1$	$-$	$Y_4 + Y_0 + \varepsilon(\beta - \alpha)(Y_x - Y_y)$
$u^\sigma \ln u$	$u^\sigma(\ln u + \alpha)$	$\pm$	$\frac{\sigma}{2}Y_4 - \frac{1}{2}Y_5 + \varepsilon Y_x + \varepsilon Y_y$
		$-$	$Y_4 + Y_0 + \varepsilon\alpha(Y_x - Y_y)$
$u^\sigma(u^\alpha(\lambda \cos \ln u +$ $\mu \sin \ln u) + \beta)$	$u^\sigma(u^\alpha(\mu \cos \ln u -$ $\lambda \sin \ln u) + \gamma)$	$+$	$\frac{1}{2}Y_5 - \left(\alpha + \frac{\sigma}{2}\right)Y_4 + Y_0 + \varepsilon a Y_x + \varepsilon b Y_y$, where $a = \alpha\beta + \gamma$, $b = \alpha\gamma - \beta$
$u^\sigma(u^\alpha + \beta \ln u)$	$u^\sigma(u^\alpha - \beta \ln u + \gamma)$	$-$	$Y_5 - (\alpha + \sigma)Y_4 + \alpha Y_0 + \varepsilon a Y_x + \varepsilon b Y_y$, where $a = \alpha\gamma - 2\beta$, $b = \alpha\gamma + 2\beta$
$u^\alpha + \mu u + \beta u^\sigma$	$u^\alpha + \mu u + \gamma u^\sigma$	$-$	$Y_4 + Y_0 + \varepsilon(\gamma - \beta)(Y_x - Y_y)$
$u^\alpha + \mu u + \beta u^\sigma$	$u^\alpha - \mu u + \gamma u^\sigma$	$-$	$(\sigma - \alpha - 1)Y_4 + Y_5 + (\alpha - 1)Y_0 + \varepsilon a Y_x + \varepsilon b Y_y$, where $a = \gamma(\alpha - 1) + \beta(\alpha - 2\sigma + 1)$, $b = \gamma(\alpha - 2\sigma + 1) + \beta(\alpha - 1)$
u^σ	0	$\pm$	$\sigma Y_4 - Y_5$
			$Y_4 - \varepsilon Y_x$
			$Y_0 - \varepsilon\kappa Y_y$
$u^\sigma(u^\alpha + \beta)$	0	$\pm$	$\frac{1}{2}Y_5 - \left(\alpha + \frac{\sigma}{2}\right)Y_4 + \varepsilon\alpha\beta Y_x$
		$-$	$Y_4 + Y_0 + \varepsilon(\gamma - \beta)(Y_x - Y_y)$
$\ln u + \alpha u^\sigma$	0	$\pm$	$\frac{\sigma}{2}Y_4 + \frac{1}{2}Y_5 - \varepsilon t\partial_x - \varepsilon\sigma\alpha Y_x$
$u^\sigma \ln u$	0	$\pm$	$\frac{\sigma}{2}Y_4 - \frac{1}{2}Y_5 + \varepsilon Y_x$

Table 6. Symmetries for equation (2), $\varphi(u) = u^{-1}$, $\psi(u) = \kappa u^{-1}$, $\kappa = \pm 1$.

$f(u)$	$g(u)$	κ	Symmetries
u^α	u^α or 0	$\pm$	$(\alpha + 1)Y_4 - Y_t$
$\ln u$	$\ln u$	$\pm$	$Y_4 - Y_t + \varepsilon t\partial_x + \varepsilon t\partial_y$
$u^\alpha(\beta \cos \ln u + \gamma \sin \ln u)$	$u^\alpha(\gamma \cos \ln u - \beta \sin \ln u)$	$+$	$(\alpha + 1)Y_4 - Y_0 - Y_t$
$u^\alpha + \gamma u^\beta$	$u^\alpha - \gamma u^\beta$	$-$	$(\alpha + \beta + 2)Y_4 + (\beta - \alpha)Y_0 - 2Y_t$
$u^\alpha + \beta \ln u$	$u^\alpha - \beta \ln u$	$-$	$(\alpha + 2)Y_4 - \alpha Y_0 - 2Y_t + 2\varepsilon\beta t(\partial_x - \partial_y)$
$\ln u$	0	$\pm$	$Y_4 - Y_t + \varepsilon t\partial_x$

Case 7. $\varphi(u) = u^\sigma$, $\psi(u) = \pm 1$. If f and g are arbitrary, a basis of Lie algebra is given by operators (10) and

$$X_8 = \varepsilon Y_5 \qquad \text{where} \quad Y_5 = \sigma x\partial_x + 2u\partial_u.$$

In table 7 the cases of its extension are given, using an auxiliary operator $Y_x = (3\sigma + 4)^{-1}\left(\frac{\sigma}{2}x^2\partial_x + 2xu\partial_u\right)$.

Case 8. $\varphi(u) = u^{-4/3}$, $\psi(u) = \pm 1$. For arbitrary functions f and g the Lie algebra is spanned by operators (10) and

$$X_8 = \varepsilon(x^2\partial_x - 3xu\partial_u) \qquad X_9 = \varepsilon Y_5 \qquad \text{where} \quad Y_5 = -\tfrac{2}{3}x\partial_x + u\partial_u.$$

Additional symmetries are presented in table 8.

Table 7. Symmetries for equation (2), $\varphi(u) = u^\sigma$, $\psi(u) = \pm 1$.

$f(u)$	$g(u)$	Symmetries
u^σ	0	$Y_4 - \varepsilon Y_x$
		$\sigma Y_4 - Y_5$
u^σ	u^α	$2\alpha Y_4 - Y_5 + \varepsilon(\sigma - 2\alpha)Y_x$
$u^\alpha + \beta u^\sigma$	$u^{\alpha-\sigma/2}$ or 0	$\left(\frac{\sigma}{2} - \alpha\right) Y_4 + \frac{1}{2} Y_5 + \varepsilon(\alpha - \sigma)\beta Y_x$
u^σ	$\ln u$	$Y_5 - \varepsilon\sigma Y_x - 2\varepsilon t\partial_y$
$u^{\sigma/2} + \alpha u^\sigma$	$\ln u$	$Y_5 - \varepsilon\sigma\alpha Y_x - 2\varepsilon t\partial_y$
$\ln u + \alpha u^\sigma$	$u^{-\sigma/2}$ or 0	$\sigma Y_4 + Y_5 - 2\varepsilon\sigma\alpha Y_x - 2\varepsilon t\partial_x$
$u^\sigma \ln u$	$u^{\sigma/2}$ or 0	$\sigma Y_4 - Y_5 + 2\varepsilon Y_x$
0	u^α	$2\alpha Y_4 - Y_5$
0	$\ln u$	$Y_5 - 2\varepsilon t\partial_y$

Table 8. Symmetries for equation (2), $\varphi(u) = u^{-4/3}$, $\psi(u) = \pm 1$.

$f(u)$	$g(u)$	Symmetries
u^α or 0	$u^{\alpha+2/3}$ or 0	$Y_5 - \left(\alpha + \frac{2}{3}\right) Y_4$
$u^{-2/3}$ or 0	$\ln u$	$Y_5 - \varepsilon t\partial_y$
$\ln u$	$u^{2/3}$ or 0	$Y_5 - \frac{2}{3} Y_4 - \varepsilon t\partial_x$

Case 9. $\varphi(u) = 1, \psi(u) = \kappa, \kappa = \pm 1$. For arbitrary functions f and g the Lie algebra is spanned by operators (10), (11) and the operators

$$X_8 = \varepsilon(4t^2\partial_t + 4tx\partial_x + 4ty\partial_y - u(4t + x^2 + \kappa y^2)\partial_u)$$
$$X_9 = \varepsilon Y_5 \qquad X_{10} = \varepsilon Y_6 \qquad X_{11} = \varepsilon Y_7 \qquad X_\infty = \varepsilon\alpha(t, x, y)\partial_u$$
$$\text{where} \qquad Y_5 = u\partial_u \qquad Y_6 = 2t\partial_x - xu\partial_u \qquad Y_7 = 2t\partial_y - \kappa yu\partial_u$$
$$\alpha_t = \alpha_{xx} + \kappa\alpha_{yy}.$$

Table 9 lists the cases of extension of this Lie algebra.

The group properties of the 1D diffusion–convection equation

$$u_t = (\varphi(u)u_x)_x + \varepsilon f(u)u_x + o(\varepsilon) \tag{12}$$

can be obtained from the above classification of 2D equation (2). The symmetries of equation (12) are presented in the rows of tables 3, 7, 8, 9 corresponding to the cases $g(u) = 0$ or such values of constant parameters that make $g(u)$ equal to zero. Besides it should be put formally $y \equiv 0$, $\partial_y \equiv 0$ in these operators.

3. Solutions of equation (2) with power-law diffusivity

Here the solutions of instantaneous source type of equation

$$u_t = (u^\sigma u_x)_x + (u^\sigma u_y)_y + \varepsilon f(u)u_x + \varepsilon g(u)u_y + o(\varepsilon) \tag{13}$$

for some forms of the convective terms $f(u)$ and $g(u)$ are sought. These are the solutions of the initial-value problem with initial function

$$u(t, x, y)|_{t=0} = (E_0 + \varepsilon E_1)\delta(x, y) \tag{14}$$

where $E_0, E_1 = \text{const}$ and $\delta(x, y)$ is a Dirac measure. When $\varepsilon = 0$ this problem for the corresponding diffusion equation has been solved in [15] (see also [16]). According to the

Table 9. Symmetries for equation (2), $\varphi(u) = 1$, $\psi(u) = \kappa, \kappa = \pm 1$.

$f(u)$	$g(u)$	κ	Symmetries
u^σ	u^σ or 0	$\pm$	$\sigma Y_4 - Y_5$
$\ln u$	$\beta \ln u$	$\pm$	$2Y_7 - 2\kappa\beta Y_6 + \varepsilon(\kappa + \beta^2)tY_0 + \frac{1}{2}\varepsilon u(y - \beta x)(\beta y + \kappa x)\partial_u$
			$u(2 - \varepsilon(x + \kappa\beta y))\partial_u$
u	u	$\pm$	$Y_4 - Y_5$
		$-$	$4\varepsilon t Y_0 - (4(x + y) + \varepsilon u(4t - x^2 + y^2))\partial_u$
			$(2\omega' - \varepsilon\omega u)\partial_u$, where $\omega = \omega(x - y)$
		$+$	$4\varepsilon t Y_0 + (4(x - y) + \varepsilon u(y^2 - x^2))\partial_u$
			$-2\varepsilon t(Y_6 + Y_7) + (4t + (x + y)^2 - \frac{1}{6}\varepsilon(x + y)^3 u)\partial_u$
			$(2 - \varepsilon(x + y)u)\partial_u$
			$(4(x + y) + \varepsilon u(4t - (x + y)^2))\partial_u$
u	0	$\pm$	$Y_4 - Y_5$
			$2\varepsilon t Y_0 + (\varepsilon xyu - 4y)\partial_u$
			$-\varepsilon t Y_6 + \left(2t + x^2 - \frac{1}{6}\varepsilon x^3 u\right)\partial_u$
			$(2 - \varepsilon xu)\partial_u$
			$(4x + \varepsilon u(2t - x^2))\partial_u$
$u + \beta e^u$	$u - \beta e^u$	$-$	$Y_4 + Y_0 + (\varepsilon(x - y)u - 2)\partial_u$
$e^u(\lambda\cos u + \mu\sin u)$	$e^u(\mu\cos u - \lambda\sin u)$	$+$	$Y_4 - Y_0 - \partial_u$
$e^{\beta u}$	$\gamma e^{\beta u}$	$\pm$	$\beta Y_4 - \partial_u$
$e^{\beta u} + \gamma e^u$	$e^{\beta u} - \gamma e^u$, $\gamma \neq 0$	$-$	$(\beta + 1)Y_4 + (1 - \beta)Y_0 - 2\partial_u$

invariance principle [17], if the problem (13), (14) is invariant under a group of transformations, then the solution should be sought among functions invariant under this group. Invariance of the initial-value problem (13), (14) means that equation (13), the manifold $t = 0$ where the initial data are given and the data themselves are invariant under the group.

All symmetries of equation (13), except for the translations, leave the initial manifold invariant. Transformations (6) change the Dirac measure to $\bar\delta = \delta - a(\xi_x + \eta_y)\delta\big|_{\substack{x=0\\y=0}} + o(a)$. Therefore, the criterion for invariance of initial condition (14) under transformations (6) takes the form

$$(\upsilon(t, x, y, u, \varepsilon) + (E_0 + \varepsilon E_1)(\xi_x + \eta_y)\delta)\Big|_{\substack{t=0, x=0, y=0\\ u=(E_0+\varepsilon E_1)\delta}} = o(\varepsilon). \tag{15}$$

Case 1. For $f(u) = u^\sigma$, $g(u) = u^\sigma$, condition (15) is satisfied by the symmetries

$$(\sigma + 1)Y_4 - Y_5 - \varepsilon Y_x - \varepsilon Y_y \qquad \text{and} \qquad Y_0 + \varepsilon Y_x - \varepsilon Y_y$$

from table 5. The solution of characteristic equations of the form $\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{dy}{\eta} = \frac{du}{\upsilon}$ for these symmetries yields the invariants

$$I_1 = t^{\frac{1}{\sigma+1}} u \left(1 + \frac{\varepsilon(x + y)}{2(\sigma + 1)}\right) \qquad I_2 = t^{-\frac{1}{2(\sigma+1)}} r \left(1 + \varepsilon\frac{\sigma(x + y)}{8(\sigma + 1)}\right)$$

$r = \sqrt{x^2 + y^2}$. Then, following the methods of the theory of approximate Lie symmetries [12], the solution is sought in the form

$$u = t^{-\frac{1}{\sigma+1}} \left(1 - \frac{\varepsilon(x + y)}{2(\sigma + 1)}\right) (U(I_2) + \varepsilon V(I_2)).$$

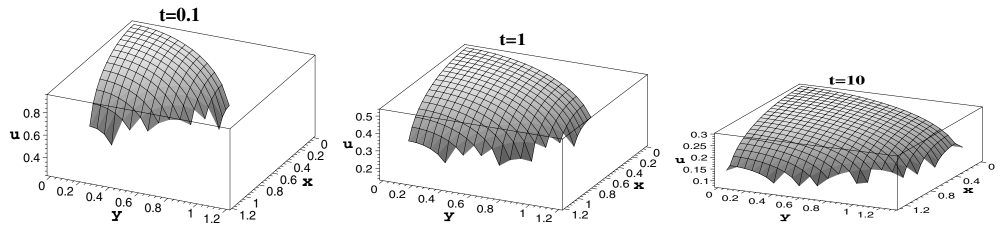

Figure 1. Solution of equation (17) when $\sigma = 3$, $E_0 = 1$, $E_1 = 1$, $\varepsilon = 0.01$.

The expansion of this function in a series in ε leads to the expression

$$u = t^{-\frac{1}{\sigma+1}} \left(\left(1 - \frac{\varepsilon(x+y)}{2(\sigma+1)} \right) U(z) + \varepsilon z \frac{\sigma(x+y)}{8(\sigma+1)} U'(z) + \varepsilon V(z) \right) + o(\varepsilon) \quad (16)$$

where $z = t^{-\frac{1}{2(\sigma+1)}} r$ is a similarity variable. The substitution of (16) into the equation

$$u_t = (u^\sigma u_x)_x + (u^\sigma u_y)_y + \varepsilon u^\sigma (u_x + u_y) + o(\varepsilon) \quad (17)$$

and splitting by powers of ε reduce (17) to the system of ordinary differential equations (ODEs)

$$(z^2 U)' + 2(\sigma+1)(zU^\sigma U')' = 0 \quad (18)$$
$$(z^2 V)' + 2(\sigma+1)(z(U^\sigma V)')' = 0$$

with the particular solution $U = \left(\frac{\sigma}{4}\left(C_1^2 - z^2\right)/(\sigma+1)\right)^{\frac{1}{\sigma}}$, $V = \left(\frac{\sigma}{4}/(\sigma+1)\right)^{\frac{1}{\sigma}} C_2 \left(C_1^2 - z^2\right)^{\frac{1}{\sigma}-1}$. Hence a solution of equation (17) is given by

$$u = \begin{cases} t^{-\frac{1}{\sigma+1}} \left(\frac{\sigma}{4}/(\sigma+1)\right)^{\frac{1}{\sigma}} \Bigg(\left(C_1^2 - r^2 t^{-\frac{1}{\sigma+1}}\right)^{\frac{1}{\sigma}} + \varepsilon \left(C_1^2 - r^2 t^{-\frac{1}{\sigma+1}}\right)^{\frac{1}{\sigma}-1} \\ \quad \times \left(C_2 - \frac{x+y}{4(\sigma+1)}\left(2C_1^2 - r^2 t^{-\frac{1}{\sigma+1}}\right)\right)\Bigg) \qquad r \leqslant C_1 t^{\frac{1}{2(\sigma+1)}} \\ 0 \qquad r > C_1 t^{\frac{1}{2(\sigma+1)}}. \end{cases}$$

The values $C_1^2 = \frac{4}{\sigma}(\sigma+1)\left(\frac{E_0}{4\pi}\right)^{\frac{\sigma}{\sigma+1}}$, $C_2 = \frac{E_1}{\sigma\pi}\left(\frac{E_0}{4\pi}\right)^{-\frac{1}{\sigma+1}}$ are defined from the condition of constant energy

$$\int_{\mathbb{R}^2} u(t, x, y)\, \mathrm{d}x\, \mathrm{d}y = E_0 + \varepsilon E_1.$$

The solution obtained is valid for finite t within the limits from 0 to the values of order ε^{-1}. Otherwise the convective terms become comparable with other terms of equation (2) and the equation takes on different group properties. Note that the values of σ running from 3 to 4 are of interest in the filtration theory. In figure 1 the solution of equation (17) is presented for $\varepsilon = 0.01$, $\sigma = 3$.

Case 2. For $f(u) = u^\sigma(u^\alpha + \beta)$, $g(u) = u^\sigma(u^\alpha + \gamma)$, $\beta, \gamma = \text{const}$, condition (15) is satisfied by the symmetries

$$X_1 = \varepsilon((\sigma+1)Y_4 - Y_5) \qquad X_2 = \varepsilon Y_0.$$

The case $\alpha = 1/2$ stands out because of the additional symmetry

$$X_3 = (\sigma+1)Y_4 - Y_5 - \varepsilon(\beta Y_x + \gamma Y_y)$$

satisfying (15). Therefore, we consider the equation

$$u_t = (u^\sigma u_x)_x + (u^\sigma u_y)_y + \varepsilon u^\sigma \left(\left(u^{\frac{1}{2}} + \beta\right)u_x + \left(u^{\frac{1}{2}} + \gamma\right)u_y\right) + o(\varepsilon). \quad (19)$$

The operator X_1 is unessential, since $X_1 = \varepsilon X_3$. The invariant test, $X_i I = o(\varepsilon)$, $i = 2, 3$, provides the invariants

$$I_1 = t^{\frac{1}{\sigma+1}} u \left(1 + \varepsilon \frac{\beta x + \gamma y}{2(\sigma + 1)}\right) \qquad I_2 = t^{-\frac{1}{2(\sigma+1)}} r \left(1 + \varepsilon\sigma \frac{\beta x + \gamma y}{8(\sigma + 1)}\right) \qquad I_3 = \varepsilon \arctan \frac{y}{x}.$$

According to [11], the solution should be sought in the form

$$u = t^{-\frac{1}{\sigma+1}} \left(\left(1 - \varepsilon \frac{\beta x + \gamma y}{2(\sigma + 1)}\right) U(z) + \varepsilon\sigma z \frac{\beta x + \gamma y}{8(\sigma + 1)} U'(z) + \varepsilon V(z, \theta)\right) \tag{20}$$

where $z = t^{-\frac{1}{2(\sigma+1)}} r$, $\theta = \arctan\frac{y}{x}$. Substitution of (20) into (19) gives the same ODE (18) in $U(z)$ and a linear PDE

$$\frac{1}{2(\sigma + 1)}(z^2 V)_z + (z(U^\sigma V)_z)_z + \frac{1}{z} U^\sigma V_{\theta\theta} + z U^{\sigma+\frac{1}{2}} U'(\sin\theta + \cos\theta) = 0 \tag{21}$$

in $V(z, \theta)$. The function $U = \left(\frac{\sigma}{4}\left(C_1^2 - z^2\right)\big/(\sigma + 1)\right)^{\frac{1}{\sigma}}$ can be taken again as a solution of (18). If we let $V = \left(\frac{\sigma}{4}\big/(\sigma + 1)\right)^{\frac{1}{\sigma}} \left(C_1^2 - z^2\right)^{\frac{1}{\sigma}-1} (C_2 - z v(\xi)(\sin\theta + \cos\theta))$, $\xi = C_1^{-2} z^2$, then (21) takes the form of a hypergeometric equation

$$\xi(\xi - 1)v'' + (\xi(2 + 1/\sigma) - 2)v' + \frac{1}{2\sigma} v - \frac{C_1^2}{2\sigma} \left(\frac{\sigma C_1^2}{4(\sigma + 1)}\right)^{\frac{1}{2\sigma}} (1 - \xi)^{\frac{1}{2\sigma}+1} = 0 \tag{22}$$

in $v(\xi)$.

When $\sigma = \frac{3}{2}$ the function $v = C_1^2\big/13\left(\frac{3}{20}C_1^2\right)^{1/3}\left(\frac{7}{3} - \xi\right)(1 - \xi)^{1/3}$ is a particular solution of equation (22) and then a solution of equation (19) is given by

$$u = \begin{cases} \left(\frac{3}{20}\right)^{2/3} t^{-2/5} \Bigg(\left(C_1^2 - r^2 t^{-2/5}\right)^{2/3} - \varepsilon \left(\frac{3}{20}\right)^{1/3} t^{-1/5} \frac{x+y}{13} \left(\frac{7}{3} C_1^2 - r^2 t^{-2/5}\right) \\ \quad + \varepsilon \left(C_1^2 - r^2 t^{-2/5}\right)^{-1/3} \left(C_2 - \frac{\beta x + \gamma y}{10} \left(2C_1^2 - r^2 t^{-2/5}\right)\right)\Bigg) & r \leqslant C_1 t^{1/5} \\ 0 \qquad r > C_1 t^{1/5} \end{cases}$$

where $C_1^2 = \frac{20}{3}\left(\frac{E_0}{4\pi}\right)^{3/5}$, $C_2 = \frac{2E_1}{3\pi}\left(\frac{E_0}{4\pi}\right)^{-2/5}$.

When $\sigma = \frac{3}{4}$ the function $v = C_1^2\big/44\left(\frac{3}{28}C_1^2\right)^{2/3}(2\ln\xi + 4\xi^2 - 13\xi - 3\xi^{-1})(1 - \xi)^{-1/3}$ is a particular solution of equation (22). When $\sigma = \frac{1}{2n}$, $n = 1, 2, \ldots$, a particular solution of (22) is defined by a polynomial of degree $n + 1$ in ξ.

Case 3. For $f(u) = u^\sigma(u^\alpha(\lambda\cos\ln u + \mu\sin\ln u) + \beta)$, $g(u) = u^\sigma(u^\alpha(\mu\cos\ln u - \lambda\sin\ln u) + \gamma)$, $\beta, \gamma = \text{const}$, the symmetries

$$X_1 = \varepsilon((\sigma + 1)Y_4 - Y_5) \qquad X_2 = \varepsilon Y_0$$

satisfy condition (15). There is an additional symmetry

$$X_3 = Y_5 - (\sigma + 1)Y_4 + 2Y_0 + \varepsilon(\beta + 2\gamma)Y_x + \varepsilon(\gamma - 2\beta)Y_y$$

subject to $\alpha = 1/2$. The invariant test, $X_i I = o(\varepsilon)$, $i = 2, 3$, provides the invariants

$$I_1 = t^{\frac{1}{\sigma+1}} u \left(1 + \varepsilon \frac{\beta x + \gamma y}{2(\sigma + 1)}\right) \qquad I_2 = t^{-\frac{1}{2(\sigma+1)}} r \left(1 + \varepsilon\sigma \frac{\beta x + \gamma y}{8(\sigma + 1)}\right)$$

$$I_3 = \varepsilon \left(\arctan\frac{y}{x} - \frac{1}{\sigma + 1} \ln t\right).$$

The group invariant solution has the same form (20), but with other similarity variables $z = t^{-\frac{1}{2(\sigma+1)}} r$, $\theta = \arctan\frac{y}{x} - \frac{1}{\sigma+1}\ln t$. Substitution of (20) into equation (13) for given $f(u)$ and $g(u)$ $(\alpha = 1/2)$ yields ODE (18) in $U(z)$ and a linear PDE

$$\frac{1}{2(\sigma+1)}(z^2V)_z + (z(U^\sigma V)_z)_z + \frac{z}{\sigma+1}V_\theta + \frac{1}{z}U^\sigma V_{\theta\theta} + zU^{\sigma+\frac{1}{2}}U'((\mu\cos\ln U - \lambda\sin\ln U)\sin\theta + (\mu\sin\ln U + \lambda\cos\ln U)\cos\theta) = 0$$

in $V(z,\theta)$.

4. Conclusion

Here the approximate Lie group analysis is applied to the 2D nonlinear diffusion–convection equation (2) with the convection taken as a small perturbation. Equation (2) inherits all symmetries X^0 of unperturbed equation (3) as 'trivial' symmetries εX^0, which are the solutions of homogeneous equation (9). We classified all functional forms of $f(u)$, $g(u)$ which admit additional 'zeroth-order' symmetries $X^0 + \varepsilon X^1$ with $X^0 \neq 0$. Such symmetries are more useful in constructing group invariant solutions. This feature is demonstrated in section 3, where the solutions of instantaneous source type of equation (13) are obtained. Two zeroth-order symmetries (case 1) allow us to reduce the initial-value problem (13), (14) to solving two ODEs. The invariance under zeroth-order symmetry and a symmetry of the form εX^0 (cases 2 and 3) reduces the problem (13), (14) to solving an ODE and a linear PDE.

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